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LETTER TO THE EDITOR

Classical non-adiabatic angles

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Abstract. If a family of tori in phase space is driven by a time-dependent Hamiltonian flow in such a way as to return after some time to the original family, there generally results a shift in the angle variables. One realisation of this process is in the cyclic *adiabatic* change of a classical Hamiltonian, and the angle change has previously been shown to separate naturally into a dynamical part and a geometrical part. Here the same geometrical angle change is extracted when the return is achieved *non-adiabatically*, and the 'dynamical' remainder calculated. Two examples are given: the precession of a spin and the rotation of phase-space ellipses.

It is known [1, 2] that the cyclic adiabatic change of an integrable Hamiltonian induces in the angle variable(s) a change $\Delta\theta$ which separates naturally into the obvious dynamical change $\Delta\theta_d$ (the time integral of the frequency), and an additional geometric change $\Delta\theta_g$. This is a classical analogue of the geometric quantum phase [3] arising naturally in the adiabatic cyclic change of a quantum Hamiltonian. As has recently been pointed out by Aharonov and Anandan [4], the same geometric part can be extracted from the phase change that occurs in a general, non-adiabatic, cyclic evolution of a quantum state, to leave a quite simple 'dynamical' remainder. Our purpose is to show that $\Delta\theta_g$ can be similarly extracted from the general, non-adiabatic, cyclic change of an action torus, with a simple remainder.

For simplicity we analyse a system with one freedom and later generalise to more. Consider an action-angle coordinate system on the phase plane, i.e. $I(q, p; X)$, $\theta(q, p; X)$ where $X = (X_1, X_2, \ldots)$ is a set of parameters with which the coordinate system can be changed. The action contours are loops (one-dimensional tori) with area $2\pi I$, and the angle is the canonically conjugate variable (whose uniform distribution is defined by the density $\delta(I - I(q, p; X))$).

The purpose of setting up this variable coordinate system is that we are now to imagine a flow in the phase space generated by a Hamiltonian $H(q, p, t)$ which causes an initial family of closed curves (tori), marked in the flow, to be carried through a cycle so as to return after time T (figure 1). At all times $0 < t < T$ there is a parameter $X(t)$ for which the curves coincide with the action contours of $I(q, p; X(t))$. This process defines a classical cyclic evolution; it is not necessary that H change slowly, or cyclically, or that the marked initial curves coincide with its contours.

Since by Liouville's theorem the area of a curve cannot change as it is transported, the action coordinate for any carried phase point is constant, $\dot{I} = 0$, and the cyclic change means $X(T) = X(0)$. In contrast, the angle variable (of a carried phase point) will generally vary in this process, and, in particular, when an initial curve has returned after time T the individual points will be shifted by an angle (the same for all points on that curve) which we now determine.

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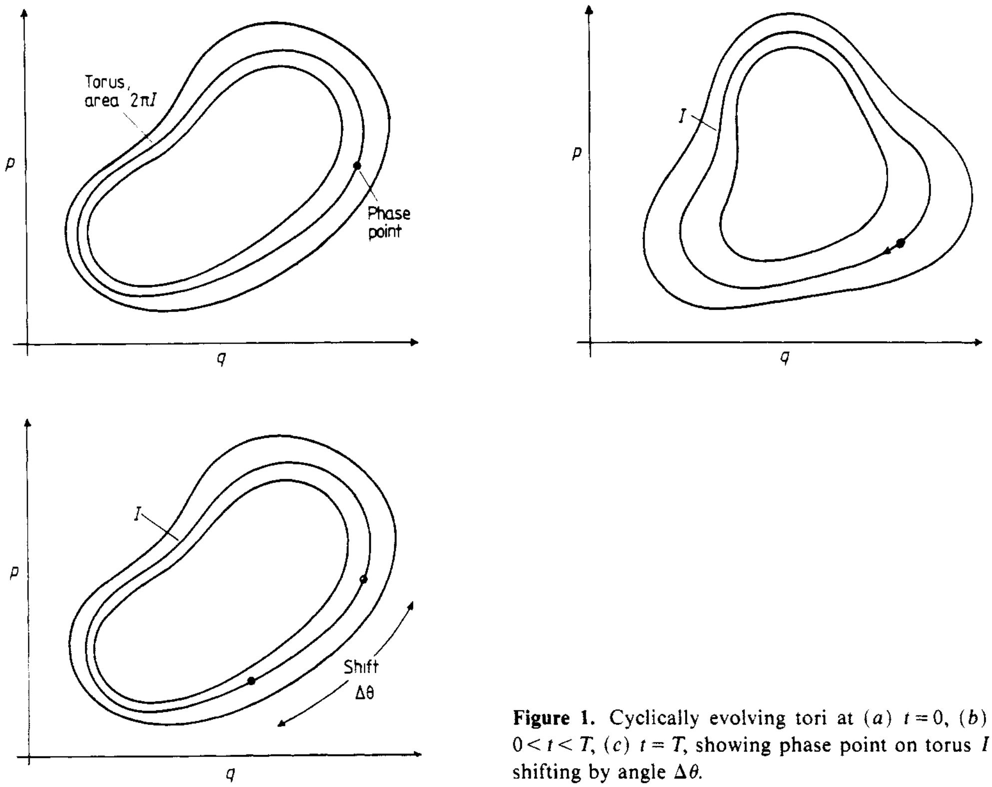

Figure 1. Cyclically evolving tori at (a) $t=0$, (b) $0<t<T$, (c) $t=T$, showing phase point on torus I shifting by angle $\Delta\theta$.

Following [1] we write the rate of change of angle of a phase point as the sum of contributions from its motion in phase space and from the changing coordinates I, θ:

$$\dot{\theta}=\partial\mathscr{H}/\partial I+\dot{X}\partial_X\theta \tag{1}$$

where

$$\mathscr{H}(\theta, I, t)\equiv H(q(\theta, I; X(t)), p(\theta, I; X(t)), t) \tag{2}$$

and $\partial_X\theta$ is the rate at which the angle at fixed q, p changes with parameters. Integrating (1) we obtain $\Delta\theta$, which does not depend on θ, as a sum of two terms that individually do depend on θ. These dependences can be eliminated by averaging round each contour of constant action; we denote this averaging by

$$\langle\ldots\rangle\equiv\int \mathrm{d}q\int \mathrm{d}p\,\delta(I-I(q,p;X))\ldots=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\ldots. \tag{3}$$

Thus we obtain

$$\Delta\theta=\Delta\theta_{\mathrm{d}}+\Delta\theta_{\mathrm{g}} \tag{4}$$

where

$$\Delta\theta_{\mathrm{d}}=\int_0^T \mathrm{d}t\langle\partial\mathscr{H}/\partial I\rangle \tag{5}$$

and

$$\Delta\theta_g = \oint \mathrm{d}X\langle\partial_X\theta\rangle = \oint \langle\mathbf{d}\theta\rangle \tag{6}$$

where $\mathbf{d}\theta$ is the angle variable derivative 1-form in parameter space.

By design, then, the angle shift has been divided into a geometric part $\Delta\theta_g$—the same as that arising naturally in the adiabatic change of a Hamiltonian [1, 2] and involving the parameter-space 1-form $\mathbf{d}\theta$—and a remaining 'dynamical' part $\Delta\theta_d$—involving not the instantaneous frequency as in the adiabatic case but its average $\langle\partial\mathscr{H}/\partial I\rangle$ round the action contour. Thus (4) is the classical analogue of Aharonov and Anandan's division [4] of a non-adiabatic quantum phase change into a geometric part occurring naturally in adiabatic change [3] and a remaining dynamical part.

Useful formulae for $\Delta\theta_g$ will now be obtained by introducing the parameter-dependent generating function of the canonical transformation from q, p to θ, I:

$$S(q, I; X) = \int_{q_0}^{q} \mathrm{d}q'\, p(q', I; X) \qquad p = \partial S/\partial q \qquad \theta = \partial S/\partial I. \tag{7}$$

We note that this allows (1) to be reinterpreted [2] as a Hamilton equation in action-angle variables: the changing X introduces a time dependence which contributes to the transformed Hamiltonian a term $\partial S/\partial t$, whose I derivative can be shown to equal the extra term $\dot{X}\partial_X\theta$ in (1) (the proof proceeds by reducing both quantities to $S_{IX} - S_{II}S_{Xq}/S_{Iq}$).

Expressing S in action-angle variables by

$$\mathscr{S}(\theta, I; X) \equiv S(q(\theta, I; X), I; X) \tag{8}$$

we have $\mathbf{d}\mathscr{S} = \mathbf{d}S + p\mathbf{d}q$ and hence in (6)

$$\langle\mathbf{d}\theta\rangle = \langle\mathbf{d}(\partial S/\partial I)\rangle = \mathbf{d}\langle(\partial\mathscr{S}/\partial I)\rangle - \frac{\partial}{\partial I}\langle p\mathbf{d}q\rangle = -\frac{\partial}{\partial I}\langle p\mathbf{d}q\rangle \tag{9}$$

where $\mathbf{d}q$ is the coordinate displacement of a torus point with fixed θ, I accompanying an infinitesimal parameter change. (The torus average $\langle\partial\mathscr{S}/\partial I\rangle$ vanishes because $\partial\mathscr{S}/\partial I$ is periodic in θ.)

Thus

$$\Delta\theta_g = -\frac{\partial}{\partial I}\oint\langle p\mathbf{d}q\rangle = -\frac{\partial}{\partial I}\left\langle\oint p\mathbf{d}q\right\rangle \equiv -\frac{\partial}{\partial I}\langle A(\theta, I)\rangle \tag{10}$$

where $A(\theta, I)$ is the phase-space area swept out during the circuit (i.e. over time T) by the point labelled θ on the torus I (figure 2). The torus average $\langle A(\theta, I)\rangle$ is independent of the X-dependent choice of origin of θ. An alternative expression is obtained by writing the first circuit integral in (10) as the flux, through the parameter-space circuit, of the 2-form $-\partial(\langle\mathbf{d}p \wedge \mathbf{d}q\rangle)/\partial I$ (cf [2]).

If the system has N freedoms, there are N actions $I = \{I_l\}$, N angles $\theta = \{\theta_l\}$ and hence N angle shifts $\Delta\theta = \{\Delta\theta_l\}$ $(1 \leq l \leq N)$. The lth dynamical and geometric shifts are given by (5) and (10) with ∂I replaced by ∂I_l and $A(\theta, I)$ replaced by the symplectic area

$$A(\theta, I) = \sum_{l=1}^{N}\oint p_l(\theta, I; X)\,\mathbf{d}q_l(\theta, I; X). \tag{11}$$

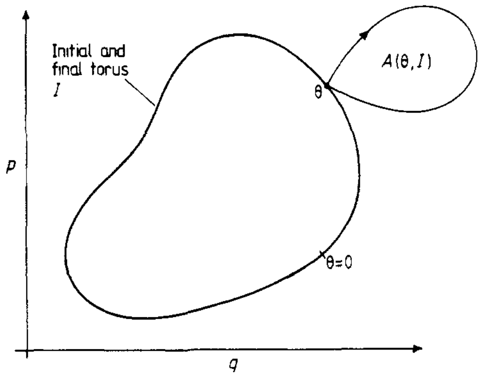

Figure 2. Area $A(\theta, I)$ of loop traced out over $0 \leq t \leq T$ by phase point labelled θ on torus I.

The form (10) for the geometric angle implies a concise expression for the semiclassical quantum phase obeying the relation [2] $\Delta\theta_g = -\hbar\partial\gamma/\partial I$. This evidently yields

$$\gamma_g = \langle A(\theta, I)\rangle/\hbar \tag{12}$$

a formula which could be rederived *ab initio* from the non-adiabatic quantum mechanics of Aharonov and Anandan [4] by using the semiclassical wavefunctions associated with moving tori (see, for example, [5]).

Our first example is the *precession of a spin* $\boldsymbol{J} = J\boldsymbol{r}$ (with unit direction $\boldsymbol{r}$) according to the law

$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \wedge \boldsymbol{r}. \tag{13}$$

The phase space is a sphere of radius J, and the flow is a rigid rotation with instantaneous angular velocity ω. This is a Hamiltonian system whose canonical variables q, p are azimuthal polar angle relative to a fixed direction $\hat{z}$ (coordinate) and J_z (momentum); the Hamiltonian is

$$H = \boldsymbol{\omega}(t) \cdot \boldsymbol{J} = J\boldsymbol{\omega}(t) \cdot \boldsymbol{r}. \tag{14}$$

The action contours are chosen to be circles of colatitude α (imagined as painted on the sphere) with direction $\boldsymbol{a}$ (called polar) as axis (figure 3). We define the action I

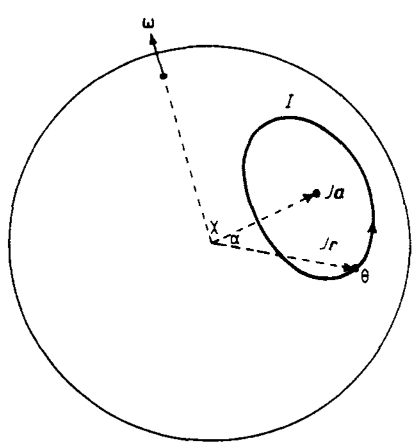

Figure 3. Geometry and notation for torus I precessing about $\boldsymbol{\omega}(t)$.

as largest when $\alpha = 0$, and therefore $1/2\pi$ times the area of the antipolar spherical cap bounded by the contour, i.e.

$$I = J(1+\cos\alpha) = J(1+\boldsymbol{a}\cdot\boldsymbol{r}). \tag{15}$$

Let $\omega(t)$ be such as to take $\boldsymbol{a}$ on a closed circuit, thereby fulfilling the conditions of our general analysis. If in addition $\boldsymbol{\omega}\cdot\boldsymbol{a} = \text{constant} \equiv \omega\cos\chi$, then (13) can be shown to model the free motion of a spinning top (the sphere) whose axle $\boldsymbol{a}$ is forcibly cycled. (Two special cases are: $\boldsymbol{\omega}$ parallel to $\boldsymbol{a}$ and changed slowly (adiabatic); and $\boldsymbol{\omega} = \text{constant}$ (simple precession).)

From (5) the dynamical angle shift is

$$\begin{aligned}\Delta\theta_{\mathrm{d}} &= J\frac{\partial}{\partial I}\int_0^T \mathrm{d}t\langle\boldsymbol{\omega}\cdot\boldsymbol{r}\rangle = J\frac{\partial}{\partial I}\int_0^T \mathrm{d}t\,\boldsymbol{\omega}\cdot\boldsymbol{a}\,\boldsymbol{r}\cdot\boldsymbol{a} \\ &= \omega T\frac{\partial}{\partial I}(I-J)\cos\chi = \omega T\cos\chi.\end{aligned} \tag{16}$$

The geometric angle shift is the solid angle Ω swept out by the axis $\boldsymbol{a}$. This was anticipated by a physical argument [1] and derived elsewhere [6, 7]. Here we obtain it from the formula (10) with the area $A(\theta, I)$ built up from individually torus-averaged elements, that is

$$\Delta\theta_{\mathrm{g}} = -\iint\frac{\partial}{\partial I}\langle\mathrm{d}A\rangle. \tag{17}$$

For $\mathrm{d}A$ it is sufficient to consider the triangle $\boldsymbol{r}$, $\boldsymbol{r}+\boldsymbol{b}\wedge\boldsymbol{r}\,\mathrm{d}\theta_b$, $\boldsymbol{r}+\boldsymbol{c}\wedge\boldsymbol{r}\,\mathrm{d}\theta_c$, where $\boldsymbol{b}\,\mathrm{d}\theta_b$ and $\boldsymbol{c}\,\mathrm{d}\theta_c$ are two infinitesimal rigid rotations of a painted action circle. The torus-averaged area (J times solid angle) of the triangle (correctly signed) is

$$\begin{aligned}\langle\mathrm{d}A\rangle &= \frac{J}{2}\mathrm{d}\theta_b\,\mathrm{d}\theta_c\int_{\text{sphere}}\mathrm{d}^2r(\boldsymbol{b}\wedge\boldsymbol{r})\wedge(\boldsymbol{c}\wedge\boldsymbol{r})\cdot(-\boldsymbol{r})\delta(\boldsymbol{a}\cdot\boldsymbol{r}+1-I/J)/2\pi \\ &= -\frac{J}{2}\mathrm{d}\theta_b\,\mathrm{d}\theta_c\int\mathrm{d}^2r(\boldsymbol{b}\wedge\boldsymbol{c})\cdot\boldsymbol{r}\delta(\boldsymbol{a}\cdot\boldsymbol{r}+1-I/J)/2\pi \\ &= -\tfrac{1}{2}\mathrm{d}\theta_b\,\mathrm{d}\theta_c[-\boldsymbol{a}\cdot(\boldsymbol{b}\wedge\boldsymbol{c})(I-J)] \\ &= -\tfrac{1}{2}\mathrm{d}\theta_b\,\mathrm{d}\theta_c(\boldsymbol{b}\wedge\boldsymbol{a})\wedge(\boldsymbol{c}\wedge\boldsymbol{a})\cdot\boldsymbol{a}(I-J).\end{aligned} \tag{18}$$

Thus $-\partial\langle\mathrm{d}A\rangle/\partial I = \mathrm{d}\Omega$ and $\Delta\theta_{\mathrm{g}} = \Omega$, as claimed.

For simple precession ($\boldsymbol{\omega} = \text{constant} \equiv \omega\hat{\boldsymbol{z}}$), $T = 2\pi/\omega$ gives a cyclic evolution and $\Omega = 2\pi(1-\cos\chi)$, so $\Delta\theta = 2\pi$, reflecting the fact that the tori have been rigidly rotated about ω, leaving points in their original positions. The quantum version of this particular case is a slight generalisation of one considered by Aharonov and Anandan [4]. We have $J = \hbar[j(j+1)]^{1/2}$ ($2j$ integer) and, for an arbitrary initial state, the following evolution generated by (14):

$$|\psi(t)\rangle = \sum_{m=-j}^{j} a_m\exp(-\mathrm{i}m\omega t)|m\rangle \tag{19}$$

where $|m\rangle$ is the eigenstate with $\langle m|J_z|m\rangle = m\hbar$. This is also cyclic for $T = 2\pi/\omega$, with total phase shift $\gamma = 2\pi j$ (up to 2π), and hence a geometric phase γ_{g} given in terms of the dynamical phase γ_d by

$$\gamma_g = 2\pi j - \gamma_d = 2\pi j + \frac{1}{\hbar}\int_0^{2\pi/\Omega} dt\langle\psi|H|\psi\rangle$$

$$= 2\pi\left(j + \sum_{m=-j}^{j} m|a_m|^2\right) = 2\pi(j + \langle\psi|J_z|\psi\rangle/\hbar). \tag{20}$$

Corresponding to the torus I is an eigenstate of the component of $\boldsymbol{J}$ along $\boldsymbol{a}$, with eigenvalue $I - j\hbar$ (= integer × $\hbar/2$), so that the expectation value in (20) is $(I = j\hbar)\cos\chi$. Thus

$$\gamma_g = 2\pi[j + (I/\hbar - j)\cos\chi] \to 2\pi[J + (I - J)\cos\chi]/\hbar \text{ as } j \to \infty. \tag{21}$$

It can be shown that $2\pi[J + (I - J)\cos\chi]$ is the torus average of the signed areas swept out by θ points during the cycle, i.e. $\langle A(\theta, I)\rangle$ in (10), so that the semiclassical relation (12) is confirmed.

In our second example the tori are *rotating ellipses* in phase space. The initial tori, with area $2\pi I$, can be written

$$I = (aq^2 + 2bqp + cp^2)/2(ac - b^2)^{1/2} \tag{22}$$

with a, b, c constant and satisfying $ac > b^2$. The Hamiltonian

$$H = \omega(p^2 + q^2)/2 \tag{23}$$

makes them rotate rigidly and non-adiabatically in $T = 2\pi/\omega$.

We find $\Delta\theta_g$ and $\Delta\theta_d$ by attaching values of θ to points moving rigidly with the tori (i.e. in circles). The transformation (7) gives, with (22),

$$\begin{aligned} q &= (2Ic)^{1/2}(ac - b^2)^{-1/4}\cos\theta \\ p &= (2I/c)^{1/2}[-b(ac - b)^{-1/4}\cos\theta + (ac - b^2)^{1/4}\sin\theta] \end{aligned} \tag{24}$$

whence averaging over θ gives

$$\langle\partial\mathscr{H}/\partial I\rangle = \frac{\omega}{2}\frac{\partial}{\partial I}\langle p^2 + q^2\rangle = \frac{\omega(a + c)}{2(ac - b^2)^{1/2}}. \tag{25}$$

Thus the dynamical angle shift (5) is

$$\Delta\theta_d = \pi(a + c)/(ac - b^2)^{1/2}. \tag{26}$$

The geometrical shift (10) involves the areas $A(\theta, I)$ (figure 2), in this case circles whose torus average is

$$\langle A(\theta, I)\rangle = \pi\langle p^2 + q^2\rangle = \pi(a + c)I/(ac - b^2)^{1/2} \tag{27}$$

so that

$$\Delta\theta_g = -\pi(a + c)/(ac - b^2)^{1/2}. \tag{28}$$

Note first that $\Delta\theta_g$ is of course the same as that calculated elsewhere [1] for an adiabatic rotation (and shown to be equal to $-\pi(\alpha + \alpha^{-1})$ where α is the axis ratio of the ellipses) and second that $\Delta\theta_d$ and $\Delta\theta_g$ cancel exactly for this rigid rotation which, as with simple precession, leaves phase points back where they started.

Note added in proof. Anandan [8] has a similar argument to ours.

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